一階正合微分方程(Exact differential equation)

➲定義

$F(x,y)=c$　且為連續函數

則滿足$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=M(x,y)dx+N(x,y)dy=0$

其中$\frac{\partial^2 F}{\partial y\partial x}=\frac{\partial^2 F}{\partial x\partial y}$　$\rightarrow$ 即$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

➲解法

由於$M(x,y)=\frac{\partial F}{\partial x}$　$\rightarrow F=\int M(x,y)\,dx+p(y)$　$p(y)$在此積分視同常數

$\frac{\partial F}{\partial y}=N(x,y)=\frac{\partial}{\partial y}\int M(x,y)\,dx+p'(y)$　$\rightarrow p'(y)=N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx$

$p(y)=\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx\right]dy$

$\therefore F(x,y)=\int M(x,y)\,dx+\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx\right]dy$

則$\int M(x,y)\,dx+\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx\right]dy=c$　即為其解

➲例題

1. $\boldsymbol{(y^2e^{xy^2}+2x^2)dx+(2xye^{xy^2}-y)dy=0}$

🕮解:

$M(x,y)=y^2e^{xy^2}+2x^2$　$\frac{\partial M}{\partial y}=2ye^{xy^2}+2xy^3e^{xy^2}$

$N(x,y)=2xye^{xy^2}-y$　$\frac{\partial N}{\partial x}=2ye^{xy^2}+2xy^3e^{xy^2}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$　為正合微分方程

$\int y^2e^{xy^2}+2x^2dx+\int\left[2xye^{xy^2}-y-\frac{\partial}{\partial y}\int y^2e^{xy^2}+2x^2dx\right]dy=c$

$\rightarrow e^{xy^2}+\frac{2}{3}x^3+\int 2xye^{xy^2}-y-2xye^{xy^2}dy=c$

$\rightarrow e^{xy^2}+\frac{2}{3}x^3-\frac{y^2}{2}=c$

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2. $\boldsymbol{(2x+y)dx+(x+y)dy=0}$

🕮解:

$M(x,y)=2x+y \quad \frac{\partial M}{\partial y}=1$

$N(x,y)=x+y \quad \frac{\partial N}{\partial x}=1$

$\rightarrow \int (2x+y)dx+\int [x+y-\frac{\partial}{\partial y}\int (2x+y)dx]dy=c$

$\rightarrow x^2+xy+\int x+y-x\,dy=c$

$\rightarrow x^2+xy+\frac{y^2}{2}=c$

➲習題

1. $(e^x+y^2)dx+(2xy+e^y)dy=0$
2. $(x^2+ysinx)dx+(y-cosx)dy=0$
3. $(x+ycosx)dx+sinxdy=0$
4. $(xsiny+e^x)dx+\left(\frac{x^2}{2}cosy+y\right)dy=0$
5. $y=(y^2-x)y'$

✍筆記欄